J. Fluid Mech. (1985), *vol.* 156, *pp.* 369–386
Printed in Great Britain

On the stability of ring modes in a trailing line vortex: the upper neutral points

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(Received 17 April 1984 and in revised form 21 January 1985)

The inviscid near-neutral stability of a trailing-vortex flow is investigated by using a normal-mode analysis in which all perturbation quantities exhibit a factor $\exp[\mathrm{i}(n\beta z-n\theta-\omega t)]$. The problem is treated as a timewise-stability problem. The dependence of the eigenvalues ω on the axial wavenumber β, which has been normalized with respect to the azimuthal wavenumber n, is found both numerically and analytically for large values of n in the upper range of values of β near $1/q$, where near-neutral modes are anticipated to occur. Here q, the swirl parameter of the flow, effectively compares the 'strengths' of the swirl and axial components of motion in the undisturbed flow. Previous normal-mode analyses based on the same form of perturbation quantities have shown that for columnar vortices the unstable modes for large values of n are ring modes, and this feature is shown to persist near the upper neutral points. In fact this work on near-neutral ring modes supplements the earlier asymptotic theory for large n, which is known to fail near $\beta = 1/q$. Our numerical and asymptotic results are in excellent agreement and are also shown to be consistent with the earlier asymptotic theory through matching. It is found that $\omega \to 0$ as $\beta \to (1/q)_-$.

1. Introduction

Much work has been done during the last 100 years on the stability of both shearing and rotating flows. In more recent years Howard & Gupta (1962) investigated the stability of inviscid flows between coaxial cylinders in which both rotation and shearing are present in the undisturbed flow. In their work the cylindrical polar (r, θ, z) velocity components $(0, V(r), W(r))$ and the pressure in the basic flow are perturbed as the result of a small disturbance and in a normal-mode analysis of the non-axisymmetric case the perturbation quantities are typified by the radial perturbation velocity

$$u(r)\exp[\mathrm{i}(n\beta z-n\theta-\omega t)]. \tag{1.1}$$

The requirement of periodicity restricts n to integer values, and real values of β then guarantee bounded solutions as $|z|\to\infty$. They reduced the linearized inviscid stability problem for such disturbances to a single differential equation in $u(r)$ with appropriate boundary conditions. Their formulation was used by Lessen, Singh & Paillet (1974), who discussed the stability of a trailing line vortex with undisturbed flow modelled on Batchelor's (1964) similarity solution for an incompressible swirling-wake flow. They used a numerical approach to solve the eigenvalue problem that arises. In terms of our somewhat different terminology they found that the modes for $\beta V > 0$ are more unstable than those for $\beta V < 0$ and this agrees with the findings in other earlier work. For these more unstable modes and the values of n considered $(1, 2, \ldots, 6)$, they showed that the maximum growth rate of the disturbance for a fixed value of the swirl parameter q, defined below in (1.3), increases with n. Their results are a useful contribution to what is really required (at least in the linearized inviscid-stability problem) – a knowledge of the dependence of ω on the parameters q, n and β. More specifically the structure of $\omega(\beta)$ is needed for the full range of values of q and n. Without loss of generality, only positive values of n will be considered.

The asymptotic analysis of Leibovich & Stewartson (1983) for $n \gg 1$ shows that in unbounded vortex flows the maximum growth rate for fixed q continues to increase with n, thereby extending the results of Lessen *et al.* (1974). This theory shows that the unstable modes are ring modes concentrated in a small neighbourhood of $r = r_0 > 0$ and that the unstable disturbances travel along the same helical paths as the undisturbed flow. They illustrated their method by applying it to the trailing line vortex, comparing their results with numerical calculations that they obtained to supplement those of Lessen *et al.* (1974) as well as those of Duck & Foster (1980). There is good agreement over β values among the various numerical results for the unstable fundamental modes whenever comparisons are possible for the values of q and n considered. All these numerical results are in good agreement with the asymptotic results at least near the maximum growth rates and the agreement improves over a wider range as n increases.

Leibovich & Stewartson (1983) were also able to find a sufficient condition for instability of a columnar vortex: with $\Omega = V/r$ and $\Gamma = rV$, the flow is unstable to inviscid disturbances if

$$V\frac{\mathrm{d}\Omega}{\mathrm{d}r}\left[\frac{\mathrm{d}\Omega}{\mathrm{d}r}\frac{\mathrm{d}\Gamma}{\mathrm{d}r}+\left(\frac{\mathrm{d}W}{\mathrm{d}r}\right)^2\right] < 0 \tag{1.2}$$

at any point of the field $(r > 0)$. For the trailing vortex, the flow usually studied numerically, we have

$$V = \frac{q}{r}(1-\mathrm{e}^{-r^2}), \quad W = \mathrm{e}^{-r^2} \quad (q > 0), \tag{1.3}$$

where q is constant. Since $V > 0$, (1.2) is equivalent to $q^2 < 2$. This is closely related to the range of validity of their asymptotic work, namely $\frac{1}{2}q < \beta < 1/q$. The asymptotic theory fails near $\beta = \frac{1}{2}q$ and $\beta = 1/q$ and the modes appear to become nearly neutral in these neighbourhoods. It is the latter neighbourhood that is considered in this paper. The mode structure near $\beta = \frac{1}{2}q$ is the subject of another investigation.

Leibovich & Stewartson point out that the full equations of Howard & Gupta (1962) have no solution with $\omega_i = 0$ and attempt to find β_c numerically such that $\omega_i(\beta) \to 0$ as $\beta \to \beta_c$. Here ω_i is the imaginary part of ω. Although the easiest disturbances to study analytically are those for $n \gg 1$, it is for small values of n that their numerical search for β_c proves to be most likely to succeed. In fact, they remark that 'the neutral modes may be more than "difficult" to compute for large n'. The reason for the difficulty is that the eigenvalues become progressively closer as ω_i decreases and consequently mode-jumping occurs in the numerical procedure as neutral conditions are approached. Furthermore, as the asymptotic theory shows, the mode separation decreases even further as n increases. The problem is so acute that Leibovich & Stewartson have obtained results for small ω_i (for various values of q) only for $n = 4$ and earlier authors present no data on the matter.

The present paper seeks to overcome the difficulties associated with mode-jumping and determine the dependence of ω on β as $\omega_i \to 0$, at least for large values of n. A scaling appropriate to the neighbourhood of $\beta = 1/q$ in §3 has the effect of separating the modes, thereby allowing a successful numerical determination of the mode structure in §4. The numerical results valid for $0 < 1-\beta q \ll 1$ suggest that for the fundamental modes $\omega \to 0$ as $\beta \to (1/q)_-$ with the same behaviour for higher modes. To help confirm our numerical results and ascertain the analytical structure of modes *very* close to neutral conditions we have carried out an asymptotic analysis in §5 for $0 < n^2(1-\beta q) \ll 1$ with $n \gg 1$. Analytical results associated with our numerical results for $0 < 1-\beta q \ll 1$ are shown through matching to be consistent with the asymptotic work of Leibovich & Stewartson (1983) for which $1-\beta q = O(1)$. Although no new results for $n = O(1)$ have been found there is some evidence to support the conjecture that $\omega \to 0$ as $\beta \to (1/q)_-$ in this case too.

2. Résumé

We consider the instability of an unbounded columnar vortex whose velocity components $[0, V(r), W(r)]$ in the undisturbed state depend only on r; $V(r)$ and $W(r)$ are positive functions such that $W \to 0$ and rV has a finite limit as $r \to \infty$. All quantities are considered to be dimensionless. This vortex is subjected to a weak non-axisymmetric disturbance and the perturbation-velocity components and pressure that result are each assumed to have a form similar to (1.1), where β and n are given positive constants with n an integer, while ω is a constant to be determined. Leibovich & Stewartson (1983) write

$$u(r) = \left(\frac{1+\beta^2 r^2}{r^3}\right)^{\frac{1}{2}} \phi(r). \tag{2.1}$$

The differential equation for ϕ is obtained from that developed for u by Howard & Gupta (1962):

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = K(r; n; \beta; \omega; q)\,\phi, \tag{2.2a}$$

where

$$K = n^2 \frac{1+\beta^2 r^2}{r^2}\left\{1 + \frac{a}{n\gamma} + \frac{b}{\gamma^2} - \frac{1+10\beta^2 r^2 - 3\beta^4 r^4}{4n^2(1+\beta^2 r^2)^3}\right\}, \tag{2.2b}$$

$$\gamma = \gamma(r) = n[\beta W(r) - \Omega(r)] - \omega \equiv n\Lambda(r) - \omega, \tag{2.2c}$$

$$a = a(r) = r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\beta r^2 W'(r) - \Gamma'(r)}{r(1+\beta^2 r^2)}\right], \tag{2.2d}$$

$$b = b(r) = -\frac{2\beta\Gamma(r)}{r(1+\beta^2 r^2)}[\beta\Gamma'(r) + W'(r)], \tag{2.2e}$$

$$\Omega(r) = \frac{V}{r}, \quad \Gamma(r) = Vr, \tag{2.2f}$$

with primes denoting differentiation with respect to r. A typical form of columnar vortex is the trailing-vortex flow in (1.3). In this case

$$b(r) = \frac{4\beta q(1-\beta q)\,\mathrm{e}^{-r^2}(1-\mathrm{e}^{-r^2})}{1+\beta^2 r^2}. \tag{2.3}$$

We wish to find the properties of ω for which non-trivial ϕ can be found to satisfy (2.2) and the conditions

$$\phi \to 0 \quad \text{as } r \to 0 \text{ and as } r \to \infty. \tag{2.4}$$

The eigenvalue ω depends on the three parameters of the flow q, n and β. The numerical studies of Lessen, Singh & Paillet (1974) show that the most unstable modes occur for $(n\beta)\,n > 0$, that is $\beta > 0$, and it is for this reason that Leibovich & Stewartson (1983) consider positive values of β and n. However, there were certain limitations in their conclusions which we wish to remove in this paper and another in preparation. In order to explain this we now briefly review their analytic procedure. In (2.2) K is a rapidly varying function since $n \gg 1$. The strategy is to find a point $r = r_0$ where K is stationary and near which it takes the form

$$K = K_0 + K_2(r-r_0)^2 + K_3(r-r_0)^3 + \dots, \tag{2.5}$$

where K_0, K_2, ... are large but independent of r. The dominant values of ϕ are now assumed to occur near $r = r_0$ so that, when (2.5) is substituted in (2.2), terms in $(r-r_0)^j$ for $j > 2$ are neglected. In this approximation ϕ may be identified with the Weber functions and solutions that decay away from the neighbourhood of $r = r_0$ are possible only if

$$K_0 K_2^{-\frac{1}{2}} = -(2s-1), \tag{2.6}$$

where s is a positive integer: $s \geqslant 1$. Since K_0 and K_2 are generally $O(n^2)$, this shows that, as a first approximation to a necessary condition for an eigensolution, K and its first derivative must vanish together (at $r = r_0$). Examination shows that the derivative of K vanishes approximately when

$$\Lambda'(r) = 0, \quad \text{that is} \quad \beta W'(r) = \Omega'(r), \tag{2.7}$$

and K is also zero then if

$$b(r) + \gamma^2 = 0, \quad \text{that is} \quad b(r) + [n\Lambda(r) - \omega]^2 = 0. \tag{2.8}$$

It is (2.7) that yields r_0 and from (2.8) ω is related to q, n and β by

$$\begin{aligned} \omega = \omega_{\mathrm{r}} + \mathrm{i}\omega_{\mathrm{i}} &= n\Lambda(r_0) \pm \mathrm{i}[b(r_0)]^{\frac{1}{2}} = n\Lambda_0 \pm \mathrm{i} b_0^{\frac{1}{2}} \\ &= n[\beta W(r_0) - \Omega(r_0)] \pm \mathrm{i}\left[\frac{2\beta\Gamma(r_0)}{r_0(1+\beta^2 r_0^2)}\right]^{\frac{1}{2}} [-\beta\Gamma'(r_0) - W'(r_0)]^{\frac{1}{2}}, \end{aligned} \tag{2.9}$$

where, for example, $b_0 = b(r_0)$. Thus the vortex is unstable to disturbances with large azimuthal wavenumber if $b_0 > 0$.

We may now discuss the limitations mentioned above. There are in fact two transitional regions which merit further study. The first arises when the value of r defined by (2.7) is very small. Then $b(r_0)$ is small by virtue of the factor $\Gamma(r_0)$ in (2.2e) coupled with the need for $V(r_0)$ to be small. A necessary and sufficient condition for this is

$$\beta \simeq \frac{\Omega'(0)}{W'(0)}, \tag{2.10}$$

which, for the trailing vortex, becomes $\beta \simeq \frac{1}{2}q$. The range of β considered by Leibovich & Stewartson (1983) in their asymptotic theory for that problem, namely $\frac{1}{2}q < \beta < 1/q$, along with $b_0 \simeq 0$ near $r = 0$, shows that this region is the neighbourhood of the lower neutral point for a specified value of q. Non-zero disturbances centred on $r = r_0$ cannot be accommodated as $r_0 \to 0$ since $\phi(0) = 0$. In a suitably refined analysis of this region it is found that the value of r_0 in (2.5) is complex. The discussion of this region is to appear at a later date.

The second transition region arises when the value of r defined by (2.7), although not small itself, implies that $b(r)$ is very small in its neighbourhood, essentially through the factor $(1-\beta q)$ in (2.3). This region, which corresponds to $\beta \to (1/q)_-$, is the neighbourhood of the upper neutral point with which we will be concerned in this paper. Here, as well as near the lower neutral point, the requirement $K = 0$ cannot be satisfied without reference to the term containing $a(r)$ in (2.2*b*). Moreover, the numerical studies of the trailing vortex by Leibovich & Stewartson (1983), like those of Duck & Foster (1980), become increasingly more difficult to complete as the upper neutral point is approached. Quite apart from the complication due to the fact that, as (2.8) clearly shows, γ vanishes at a point in the complex r-plane close to the real axis, the phenomenon of mode-jumping assumes serious proportions. It is clear from (2.6) that there are many modes for fixed β and it was shown by Leibovich & Stewartson (1983) that the separation in ω_1 between adjacent modes is $O(n^{-\frac{1}{2}}b_0^{\frac{1}{4}})$. Thus the possibility of the solution jumping to an adjacent mode as the numerical analyst attempts to trace a mode as β changes is always a matter for concern and it becomes even more serious as $b_0 \to 0$. Indeed all the computations so far reported had to be terminated before the neutral points were reached. In §3 we develop a more refined asymptotic theory aimed at allowing us to separate satisfactorily the various modes for $n \gg 1$ as the upper neutral point is approached.

3. Upper-neutral-point analysis

In order to study the upper neutral point we need to establish stretchings of the variables and parameters for $n \gg 1$ that are appropriate to the neighbourhood of the neutral point. This is achieved through an order-of-magnitude argument which also reveals the limiting processes underlying the asymptotic analysis. To be definite we assume that W' and Ω' are negative for $r > 0$, that $\beta > \Omega'(0)/W'(0)$ and that (2.7) has at least one real positive root, say at $r = r_0$. This is certainly the case for the trailing vortex and is generally to be expected if the circulation Γ is constant far from the axis while the axial velocity is exponentially small there. The asymptotic theory of Leibovich & Stewartson (1983) for $n \gg 1$ breaks down when $b(r)$ becomes sufficiently small to prohibit ignoring the term containing $a(r)$ in (2.2*b*). Then for values of r sufficiently close to r_0,

$$\frac{a}{n\gamma} \sim \frac{b}{\gamma^2},$$

while (2.8) shows $\gamma \sim b^{\frac{1}{2}}$. Since $a(r)$ is $O(1)$ near r_0, this shows that, near the upper neutral point, $\gamma \sim n^{-1}$ and $b \sim n^{-2}$.

It then follows from (2.3) that the limit process for this inner region of the β domain is described by

$$n \to \infty, \quad \beta \to \left(\frac{1}{q}\right)_-, \quad n^2(1-\beta q) = O(1). \tag{3.1}$$

In the usual way we are led to write

$$b(r_0) = b_0 = \frac{B_0}{n^2}, \tag{3.2}$$

where B_0 is a positive quantity which remains finite under the inner-limit process (3.1). Again, remembering that $\Lambda'(r_0) = 0$, we write

$$\gamma(r) = \frac{G_0}{n} + \tfrac{1}{2}n\Lambda''(r_0)\,(r-r_0)^2 + O(n[r-r_0]^3), \tag{3.3}$$

where $$G_0 = n^2\Lambda(r_0) - n\omega. \tag{3.4}$$

Then G_0 remains finite under the inner-limit process. It follows from (3.3) that $r-r_0$ is $O(n^{-1})$ and we accordingly introduce the new independent variable τ defined by

$$r = r_0 + \frac{\tau}{n}, \tag{3.5}$$

where τ is $O(1)$. Substitution of (3.5) into (2.2a) leads to the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2} = \frac{1+\beta^2 r_0^2}{r_0^2}\left[1 + \frac{a_0}{T(\tau)} + \frac{B_0}{[T(\tau)]^2}\right]\phi, \tag{3.6}$$

where $$T(\tau) = G_0 + \tfrac{1}{2}\Lambda''(r_0)\,\tau^2, \tag{3.7}$$

the error being $O(\phi/n)$. A simpler form of (3.6) results from writing

$$G_0 = B_0^{\frac{1}{2}} p, \tag{3.8a}$$

$$\tau = 2^{\frac{1}{2}} B_0^{\frac{1}{4}} [\Lambda''(r_0)]^{-\frac{1}{2}} \zeta, \tag{3.8b}$$

$$C = 2^{\frac{1}{2}} B_0^{\frac{1}{4}} [\Lambda''(r_0)]^{-\frac{1}{2}} (1+\beta^2 r_0^2)^{\frac{1}{2}} r_0^{-1}, \tag{3.8c}$$

$$a_0 = \frac{A r_0^2 \Lambda''(r_0)}{2(1+\beta^2 r_0^2)} = \frac{A B_0^{\frac{1}{2}}}{C^2}. \tag{3.8d}$$

We find $$\frac{\mathrm{d}^2\phi}{\mathrm{d}\zeta^2} = C^2\left[1 + \frac{A}{C^2(\zeta^2+p)} + \frac{1}{(\zeta^2+p)^2}\right]\phi. \tag{3.9}$$

For the trailing vortex $\Lambda''(r_0) > 0$ and ζ is real. For this flow, with the approximation $\beta \approx 1/q$, it follows from (2.7) that r_0 is the positive root of the equation

$$\mathrm{e}^{r^2} = 1 + r^4 + \frac{r^2}{q^2} \quad \text{with } q^2 < 2, \tag{3.10a}$$

$$a_0 = 4qr_0^2\,\mathrm{e}^{-r_0^2}, \tag{3.10b}$$

$$b_0 = \frac{4q^2(1-\beta q)\,(1-\mathrm{e}^{-r_0^2})\,\mathrm{e}^{-r_0^2}}{q^2+r_0^2}, \tag{3.10c}$$

and from (3.8) and (3.10b)

$$A = \frac{2(q^2+r_0^2)}{q^2+r_0^2-2}. \tag{3.10d}$$

Furthermore, $$C = \left[\frac{n^2(1-\beta q)\,(q^2+r_0^2)^2}{q^2 r_0^2 (q^2+r_0^2-2)^2}\right]^{\frac{1}{4}}, \tag{3.10e}$$

where, as in (3.10c), the factor $(1-\beta q)$ has been retained. From (3.10a) we see that, as q^2 increases from small positive values to the value 2, r_0 decreases from large values to zero. It follows that $A > 0$ for $0 < q^2 < 2$; moreover, as $q^2 \to 2_-$, $A \to \infty$ and the theory breaks down. Henceforth we consider the trailing-vortex-flow model only. Then as $|\zeta| \to \infty$, the coefficient of ϕ in (3.9) tends to a positive constant C^2 and in order that the properties of ϕ reflect our assumption that the dominant part of the solution is confined to the neighbourhood of $r = r_0$, it is necessary to demand that ϕ is exponentially small as $\zeta \to \pm\infty$. We are thus faced with an eigenvalue problem from which p can be found as a function of the two parameters A and C. Whereas the problem associated with (2.2a) implies a dependence of ω on n as well as the two parameters q and β, there is no dependence of p on n implied by (3.9) since it has been obtained from (2.2) through stretching transformations and by retaining only the coefficients of terms of highest order in n.

Now under the limit process (3.1) we see that A and C are each $O(1)$. Of course it is quite in order to consider intermediate and outer-limit processes in which $C \to \infty$ as $n \to \infty$ and this we will soon do. In a related way, although in the refinement associated with (3.1) we have seen that $r - r_0$ is $O(n^{-1})$, when β decreases from $1/q$ with $1 - \beta q$ increasing from $O(n^{-2})$ through intermediate orders until it is $O(1)$, we find $r - r_0$ has increased to $O(n^{-\frac{1}{2}})$ and this is confirmed by the analysis of Leibovich & Stewartson (1983). When $1 - \beta q$ and $r - r_0$ are of these respective larger orders, consideration of the orders of magnitude in (3.5) and (3.8) reveals that $\zeta = O(C^{-\frac{1}{4}})$. Thus, when C is large and positive, (3.9) may be solved by writing $\zeta = \eta / C^{\frac{1}{4}}$ and expanding both ϕ and p as series in descending powers of C, namely

$$\phi = \sum_{k=0}^{\infty} \frac{\phi_k(\eta)}{C^k} \quad \text{and} \quad p = \sum_{k=0}^{\infty} \frac{p_k}{C^k}. \tag{3.11}$$

When these forms are substituted into (3.9) it follows immediately that $p_0^2 = -1$. We choose $p_0 = -\mathrm{i}$ for unstable modes. Not only is this necessary for consistency here but also for consistency with the Leibovich–Stewartson results with which a match is essential, though expected to be automatic. This matter is referred to again in §6. The equation for ϕ_k $(k \geqslant 0)$ is

$$\frac{\mathrm{d}^2\phi_k}{\mathrm{d}\eta^2} = \phi_k[2\mathrm{i}p_1 + 2\mathrm{i}\eta^2] + 2\mathrm{i}p_{k+1} H_k + L_k, \tag{3.12}$$

where $H_0 = 0$, $L_0 = 0$; for $k \geqslant 1$, $H_k = 1$ and L_k is a series in $\phi_j(\eta)$ with $0 \leqslant j \leqslant k-1$ whose coefficients are polynomials in p_j and η^2. For consistency with the vanishing of ϕ at infinity, noting that $|\eta| \to \infty$ as $|\zeta| \to \infty$, we have

$$p_1 = -\tfrac{1}{2}(2s-1)(1-\mathrm{i}), \tag{3.13}$$

where, as before, s takes all positive integer values. Then ϕ_0 is a Weber function and in particular

$$\phi_0 = \exp[-\tfrac{1}{2}(1+\mathrm{i})\eta^2] \quad \text{if } s = 1, \tag{3.14}$$

$$\phi_0 = \eta \exp[-\tfrac{1}{2}(1+\mathrm{i})\eta^2] \quad \text{if } s = 2. \tag{3.15}$$

The computations of p_k and ϕ_k may now be carried out progressively and at each stage p_k is fixed by the consistency requirement that ϕ vanish at infinity. We find in particular

$$p_2 = \tfrac{9}{16} - \tfrac{1}{2}A \quad \text{if } s = 1, \tag{3.16}$$

$$p_2 = \tfrac{45}{16} - \tfrac{1}{2}A \quad \text{if } s = 2. \tag{3.17}$$

These correspond respectively to the first even and first odd eigensolutions. An important issue here is the separation of the many modes over a range of values of C. As remarked above, when B_0 is sufficiently large and hence C, a match with the results in §2 is achieved since $p_0 = -\mathrm{i}$ leads again to $\omega_\mathrm{i} \sim b_0^{\frac{1}{2}}$. The mode separation in ω_i is $O(b_0^{\frac{1}{4}} n^{-\frac{1}{2}})$ but the difficulty in distinguishing between modes as $b_0 \to 0$ is averted by using (3.9) and the associated results. This is because at a fixed value of A, as b_0 decreases, C also decreases and correspondingly, as determined by the term $(p_1)_\mathrm{i}/C$ in (3.11), the mode separation in p_i improves. Thus we anticipate that the properties of the modes as C decreases over a suitable range of values will be readily found by solving (3.9) numerically under suitable conditions at infinity. Although the numerical work of Leibovich & Stewartson (1983) suggests that the neutral point for specified A (or q) may occur at a positive (specifically non-zero) value of C, their results are not expected to be reliable in this region. For the reasons stated the numerical results based on the solution of (3.9) as described in §4 are expected to be more reliable. Moreover, application of the standard procedure in Rayleigh's theorem to (3.9) leads to the result

$$p_i \int_{-\infty}^{\infty} |\phi|^2 \frac{[(\zeta^2+p_r)^2+p_i^2]+2C^2A^{-1}(\zeta^2+p_r)}{[(\zeta^2+p_r)^2+p_i^2]^2}\, \mathrm{d}\zeta = 0. \tag{3.18}$$

Now C^2A^{-1} may be scaled out of this equation by writing $\zeta^2 = \lambda^2C^2A^{-1}$ and $p = \rho C^2A^{-1}$. Then, for $p_i \neq 0$, the integral that results must vanish. Thus for $C \rightarrow 0_+$ we must have that p is $O(C^2A^{-1})$ and in particular p_r must be negative and precisely of this order, while p_i may well be of even lower order. In fact, as completing the square shows,

$$|p_i| < C^2A^{-1}. \tag{3.19}$$

It is therefore not unreasonable to expect even more strongly that the neutral points of the modes we have been discussing will all occur at positive values of C which may nevertheless be quite small. All these conclusions from (3.18) are confirmed by the numerical results and analysis described in the sections that follow except that the neutral points are found *not* to occur at positive values of C but rather as $C \rightarrow 0_+$.

4. Numerical determination of the upper near-neutral modes

For a given trailing-vortex flow, q is known and A is found from (3.10*d*). With A fixed in this way, p depends on only one parameter C. Our object then is to supplement the work of Leibovich & Stewartson (1983) by computing the eigenvalues in (3.9) for progressively smaller values of C. An attempt to approach the critical value $C = C_c$ such that $p_i \rightarrow 0$ as $C \rightarrow C_c$ leads to a result which, in view of the previously published numerical work, and the expectation that $C_c > 0$, is very interesting. In fact it is found that no neutral mode is reached before C becomes zero. With A fixed, both even and odd eigensolutions are considered for each value of C; for the former $\phi(0)$, and for the latter $\dot{\phi}(0)$ is normalized to unity, where $\dot{\phi}(\zeta) = \mathrm{d}\phi/\mathrm{d}\zeta$. For each, the condition that ϕ be exponentially small as $\zeta \rightarrow \infty$ is incorporated in the numerical procedure by writing

$$\chi_N + C\phi_N = \chi(\zeta_N) + C\phi(\zeta_N) = 0, \tag{4.1}$$

where the substitution $\chi = \dot{\phi}$ is used in writing (3.9) as a system of first-order differential equations and ζ_N is the terminal point in the numerical process. Then for fixed values of C and p the system is approximated by using the trapezium rule in the manner described by Cebeci & Bradshaw (1977) under the name of the box method. Gaussian elimination efficiently reduces the coefficient matrix (a band matrix of bandwidth 5) to upper triangular form. Integration along the real-ζ-axis is satisfactory except for marginally unstable modes. For these the singularity at $\zeta = (-p)^{\frac{1}{2}}$ in (3.9) lies close to the real axis and in the first quadrant of the ζ-plane. For such modes the contour of integration must be deformed to lie below the real-ζ-axis. A contour of suitable though somewhat elaborate form is given by

$$\zeta = \xi - \mathrm{i}\kappa(1+p_i)\,\xi \exp[-\mathrm{Re}\{(\xi^2+p)^2\}], \tag{4.2}$$

where $\xi = \mathrm{Re}\,(\zeta)$ and κ is a positive contour control factor. As $|p_i|$ decreases towards zero (in the procedure described below) the contour shape is further controlled by the factor $(1+p_i)$, which remains positive. Values of κ between 2.0 and 3.5 produce contours which, for our purposes, do not approach too closely the singularity in the first or third quadrant where ζ^2+p vanishes. A non-uniform grid along the ξ-axis is used. Typically we use eight step sizes ranging from 0.00125 to 10, the former where the solution changes rapidly as the contour skirts the singularity, the latter for large values of ξ. A check on convergence using successive halving of step size shows that, for all the values of A and C considered (except possibly for quite small values of C), three-figure accuracy in ϕ and $\dot{\phi}$ is guaranteed when 418 grid points are used. It is further found that if ζ_N is given different values in the range $60 < \zeta_N < 90$ this accuracy is retained. The value we used most commonly is $\zeta_N = 67$.

For a fixed value of C, Newton's method is used in an iteration on p leading to a final estimate that results in values of $\dot{\phi}(0)$ and $\phi(0)$ (for even and odd eigensolutions respectively) of typical size 10^{-6}. To initiate the Newton procedure at a starting value of C, an adequate approximation to the eigenvalue is obtained from (3.11) (which is appropriate for large values of C) by truncating the series for p after two terms; for example, for the first even eigensolution, $s = 1$ and the initial estimate is

$$-\mathrm{i}-\frac{1-\mathrm{i}}{2C}. \tag{4.3}$$

The corrected value obtained from this is used as the initial estimate of p for a smaller neighbouring value of C and the corresponding accurate value is found. For decreasing values of C the continuation method, using linear extrapolation, then provides successive initial estimates for the p-iterations. Added confidence in our findings has been gained through continual checks on detailed results for both the iteration on p and the eigenfunction ϕ for various values of C. Typically, for the monitored step lengths in C used, up to three or four iterations are needed to produce the small values of $\dot{\phi}(0)$ or $\phi(0)$ mentioned above.

A FORTRAN program, written in complex mode, deals with both the solution of the equation and the continuation process for a specified value of A. A convenient test case for each value of s considered ($s = 1, 2, 3$) is $A = 1$. Although this value corresponds to a complex value of q it leads to results with the same essential features as those for physically meaningful flows (see figures 1 and 2). Thus, using $C = 5$ as a starting value in this test case, we have as an initial approximation $p \approx -0.1-0.9\mathrm{i}$ for the first even eigensolution, corresponding to $s = 1$. Our most extensive results are in fact for the first even eigensolution and these have been obtained for $q = 0.4$, 0.8, 1.0, 1.2 as well as for the test case $A = 1$. The program was also run for $s = 2, 3$ but only with $A = 1$.

For the step lengths in C used, no evidence of mode-jumping appears. However, to maintain accuracy, it is necessary to reduce the step length $(-\Delta C)$ progressively for smaller values of C. In the test case, for example, $\Delta C = -0.05$ at $C = 0.65$ and $\Delta C = -0.0125$ at $C = 0.30$. It is also necessary to transfer to double precision; in the test case this was done for $C < 1$. Indeed it is found that as $C \to 0_+$ the singularity at $\zeta = (-p)^{\frac{1}{2}}$ moves very close to the origin of ζ, the initial point on the contour of integration. It is only then that the numerical results eventually become unreliable. Typical features of the numerical results appear clearly in the various tables and graphs we have prepared, some of which are presented in this paper. Discussion of the significance of the numerical work is deferred briefly.

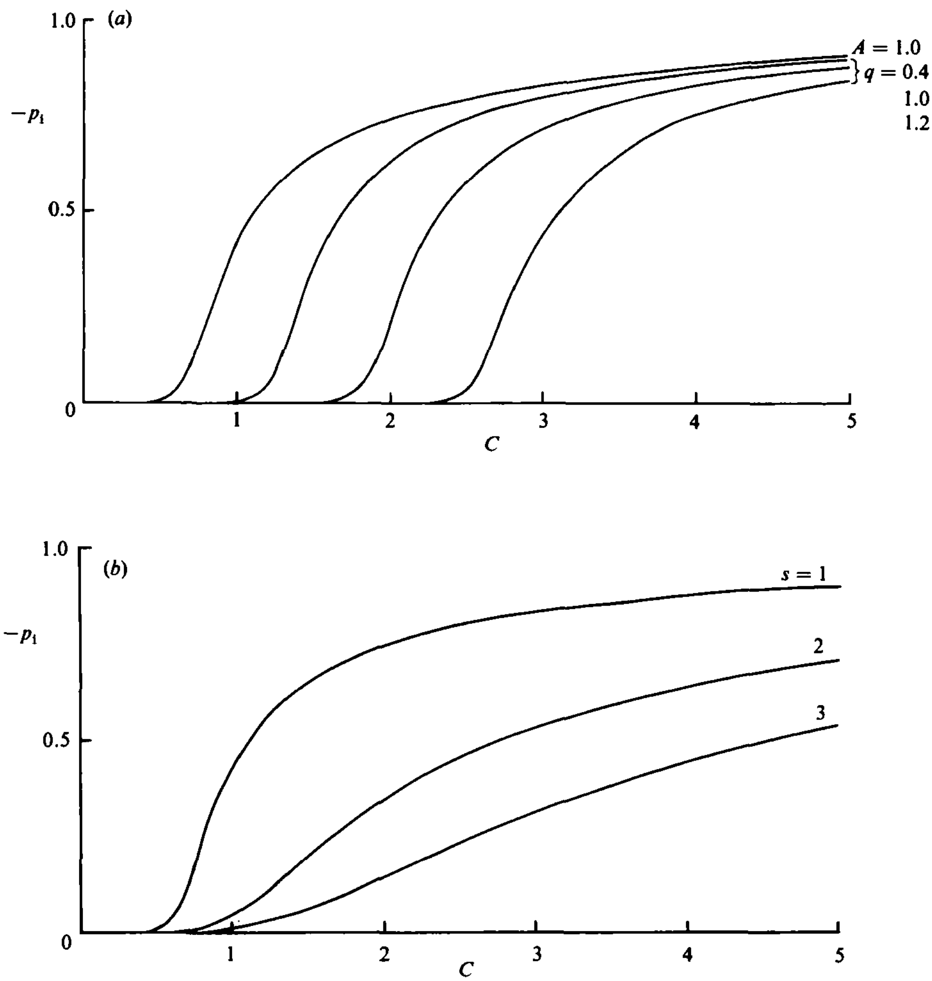

FIGURE 1. Scaled growth rates $-p_i$ for $n \gg 1$ for the trailing vortex computed numerically as functions of $C \propto n^{\frac{1}{4}}(1-\beta q)^{\frac{1}{4}}$ by the method of §4: (*a*) curves for fundamental modes ($s = 1$) for $A = 1$ (q complex) and $q = 0.4, 1.0, 1.2$ as shown; (*b*) curves for the first three modes ($s = 1, 2, 3$) for $A = 1$.

5. Stability analysis very near the upper neutral point

The numerical results of the previous section, as displayed in figures 1 and 2, indicate that an asymptotic analysis should be developed to reveal the analytic dependence of p on C as $C \to 0_+$. The conclusion from (3.18) that p is $O(C^2A^{-1})$ also indicates this. We now describe such an analysis with special attention to the case of the first even eigensolution; an asymptotic formula relating p to C is obtained for this case. A similar formula associated with the first odd eigensolution has also been found. The second even eigensolution is considered briefly.

The singularity in (3.9) is dealt with by first writing $p = -P$ and $\zeta = P^{\frac{1}{2}}t$; the equation becomes

$$\frac{d^2\phi}{dt^2} = \phi\left[C^2P + \frac{A}{t^2-1} + \frac{C^2}{P(t^2-1)^2}\right]. \tag{5.1}$$

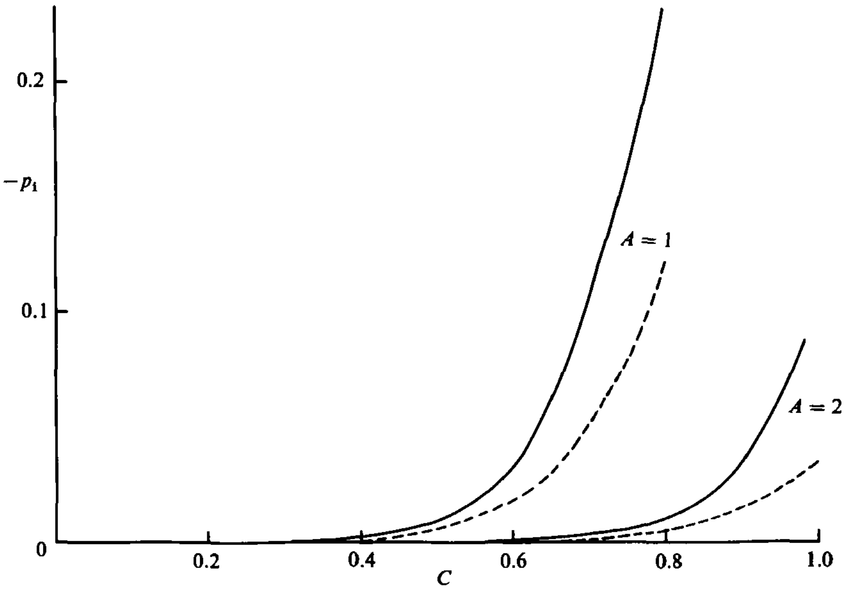

FIGURE 2. Comparison of numerically computed (——) and asymptotic (---) scaled growth rates $-p_i$ for $n \gg 1$ for small values of C with $A = 1$ for the upper pair of curves and $A = 2$ for the lower pair.

Since the evidence is that $|p|$ is small when C is small, the solution ϕ is being considered in a small region in which $\zeta = O(|p|^{\frac{1}{2}})$. The behaviour of ϕ near $t = \pm 1$ suggests writing

$$\phi = \psi(t)\,(1-t^2)^{-\mu}, \tag{5.2}$$

where

$$C^2 = 4\mu(\mu+1)\,P. \tag{5.3}$$

We then obtain

$$(1-t^2)\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2}+4\mu t\frac{\mathrm{d}\psi}{\mathrm{d}t} = \psi[(1-t^2)\,C^2P-A-2\mu+4\mu(\mu+1)]. \tag{5.4}$$

Ignoring C^2P and writing $Z = t^2$ we convert this to the hypergeometric equation

$$Z(1-Z)\frac{\mathrm{d}^2\psi}{\mathrm{d}Z^2}+[\tfrac{1}{2}-(\tfrac{3}{2}-2\mu)\,Z]\frac{\mathrm{d}\psi}{\mathrm{d}z}-(\mu^2+\tfrac{1}{2}\mu-\tfrac{1}{4}A)\,\psi = 0, \tag{5.5}$$

the properties of which are listed by Abramowitz & Stegun (1968). The solution of (5.5) leading to even eigensolutions ϕ is the hypergeometric function $F(a, b; c; Z)$. Choosing $b > a$ we have

$$\psi = F(-\tfrac{1}{2}N-\mu-\tfrac{1}{4}, \tfrac{1}{2}N-\mu-\tfrac{1}{4}; \tfrac{1}{2}; Z), \tag{5.6}$$

where

$$N = (A+\tfrac{1}{4})^{\frac{1}{2}}. \tag{5.7}$$

The 'inner' solution ϕ in (5.2) obtained in this way is expected to match (as $t \to \infty$) with terms in an 'outer' solution and in anticipation of this we find for $|t^2| \gg 1$ and $|\arg(-t^2)| < \pi$

$$\phi = (-t^2)^{-\mu}\,(1-t^{-2})^{-\mu}\,\{c_0(-t^2)^{m_1}\Sigma_1+d_0(-t^2)^{m_2}\Sigma_2\}, \tag{5.8}$$

where

$$m_1 = \tfrac{1}{2}N+\mu+\tfrac{1}{4}, \quad m_2 = -\tfrac{1}{2}N+\mu+\tfrac{1}{4}$$

and Σ_1 and Σ_2 are power series in t^{-2}, each with leading-term unity. Again

$$c_0 = \frac{\pi^{\frac{1}{2}}(N-1)!}{[\frac{1}{2}N-(\mu+\frac{5}{4})]!\,[\frac{1}{2}N+(\mu-\frac{1}{4})]!}, \quad d_0 = \frac{\pi^{\frac{1}{2}}(-N-1)!}{[-\frac{1}{2}N-(\mu+\frac{5}{4})]!\,[-\frac{1}{2}N+(\mu-\frac{1}{4})]!}. \tag{5.9}$$

We turn our attention now to the 'outer' region, for which we write $C\zeta = O(1)$ with $\phi \to 0$ as $|\zeta| \to \infty$. Then (3.9) leads to

$$\frac{d^2\phi}{d\zeta^2} = \phi\left[C^2 + \frac{A}{\zeta^2}\right]. \tag{5.10}$$

The solution that decays at infinity is

$$\zeta^{\frac{1}{2}}[I_N(C\zeta) - I_{-N}(C\zeta)], \tag{5.11}$$

and when the modified Bessel functions are expressed as ascending series (Abramowitz & Stegun), we find

$$\phi(\zeta) = \zeta^{N+\frac{1}{2}} \sum_{k=0}^{\infty} C_k \zeta^{2k} + \zeta^{-N+\frac{1}{2}} \sum_{k=0}^{\infty} D_k \zeta^{2k}, \tag{5.12}$$

where C_k and D_k are known. When only the leading term in each series is considered, the terms obtained,

$$C_0\zeta^{N+\frac{1}{2}} + D_0\zeta^{-N+\frac{1}{2}} \quad \text{with } \frac{C_0}{D_0} = -\left(\frac{C}{2}\right)^{2N}\frac{(-N)!}{N!}, \tag{5.13}$$

can be matched with similarly retained terms in (5.8) when these are expressed in terms of ζ:

$$c_0\left(-\frac{\zeta^2}{P}\right)^{\frac{1}{2}N+\frac{1}{4}} + d_0\left(-\frac{\zeta^2}{P}\right)^{-\frac{1}{2}N+\frac{1}{4}}. \tag{5.14}$$

The matching *powers* have already led to (5.13) and (5.14); to complete the match we require

$$\frac{C_0}{D_0} = \frac{c_0}{d_0}\left(-\frac{1}{P}\right)^N. \tag{5.15}$$

Now $N > 0$ and $|P| \ll 1$ so that to leading order $c_0 = 0$. Thus from (5.9), since by choice $\mu > 0$, we must have $\frac{1}{2}N - (\mu + \frac{1}{4})$ equal to zero or a negative integer, $-m$ say; then

$$\mu \approx \mu_0 = \tfrac{1}{2}N + m - \tfrac{1}{4} \quad (m \geqslant 0), \tag{5.16}$$

and F reduces to a polynomial of degree m. Let

$$\mu = \mu_0 + \epsilon = \mu_0 + \epsilon_r + i\epsilon_i, \tag{5.17}$$

where $|\epsilon| \ll 1$. In terms of C, the gauge parameter in this section, it turns out that $\epsilon_r = O(C^4)$ and $\epsilon_i = O(C^{4N})$. The leading-order contribution to P, as obtained from (5.3), is

$$P_0 = \frac{C^2}{4\mu_0(\mu_0+1)}, \tag{5.18}$$

which is real. The leading-order contribution to P_i may be extracted from the matching condition (5.15), which yields

$$\epsilon = (\tfrac{1}{4}C^2)^N \frac{P_0^N\, e^{-iN\pi}\, \pi(N+m-\frac{1}{2})!\,(N+m)!}{\sin N\pi\, [N!\,(N-1)!]^2\, m!\,(m-\frac{1}{2})!}, \tag{5.19}$$

where we have used the result

$$\left(-\frac{1}{P}\right)^N \approx \frac{e^{iN\pi}}{P_0^N}, \tag{5.20}$$

based on the assumption, which turns out to be consistent as will be seen below, that P (and hence ω) has a small positive imaginary part. Since the term ignored in (5.1), namely C^2P, is $O(C^4)$ it will lead to a contribution to ϵ in (5.17), the real part of which is $O(C^4)$, and this is larger than the order, $O(C^{4N})$, in (5.19). Thus only the imaginary part of (5.19) is relevant. Hence

$$\epsilon_i = -(\tfrac{1}{4}C^2)^N \frac{P_0^N\,\pi(N+m-\frac{1}{2})!\,(N+m)!}{[N!\,(N-1)!]^2\,m!\,(m-\frac{1}{2})!}. \tag{5.21}$$

From (5.21), (5.17), (5.3) and the fact that $p = -P$, we obtain

$$p_i = -\frac{\pi(N+m-\frac{1}{2})!\,(N+m)!\,(2\mu_0+1)\,C^{4n+2}}{4^{2N+1}[N!\,(N-1)!]^2\,m!\,(m-\frac{1}{2})!\,[\mu_0(\mu_0+1)]^{N+2}}. \tag{5.22}$$

This is consistent with our assumption above and also with the sign of p_i in our numerical results, namely $p_i < 0$. If p_i in (5.22) had turned out to be positive, the result would have been inconsistent with the assumption under which it was derived and we should then have concluded that (3.9) has no eigensolutions for sufficiently small values of C. This would imply that for $n \gg 1$ the (even) modes of the problem in (2.2) and (2.4) cease to exist before β attains the value $1/q$. However, the results of this section show that this is not the case and that for all even modes $\omega_i \to 0_+$ as $\beta \to (1/q)_-$ at the upper neutral point when n is large and the same is found to hold for the odd modes.

In the case $m = 0$ (that is for the first even mode), ϵ_r is related to C by writing

$$\psi = \psi_0 + C^4\psi_1 + o(C^4), \tag{5.23}$$

with $\psi_0 = 1$ and retaining the term $O(C^4)$ in (5.4), namely $C^2P \approx C^2P_0$. In solving the resulting first-order equation in $d\psi_1/dt$, we demand that $d\psi_1/dt$ is an odd function of t and is proportional to t as $t \to \infty$. The correction is continued analytically through $t = 1$ by use of the relation

$$(1-t^2)^\nu = \begin{cases} |1-t^2|^\nu & (t<1), \\ |t^2-1|^\nu\,e^{i\nu\pi} & (t>1) \end{cases} \tag{5.24}$$

for any constant ν. In fact (5.24) is equivalent to (5.20) and is derived on the same assumption, namely that P has a small positive imaginary part. For the lowest mode, this approach yields

$$C^2P_0 = -\frac{\epsilon_r(16\mu_0^2-1)}{2\mu_0} \tag{5.25}$$

and, after use is made of (5.3), the formula for p in this mode is found to be

$$p = -P \approx -\frac{C^2}{4\mu_0(\mu_0+1)} - \frac{(2\mu_0+1)\,C^6}{8\mu_0^2(16\mu_0^2-1)\,(\mu_0+1)^3}$$
$$-i\frac{\pi(N-\frac{1}{2})!\,N!\,(2\mu_0+1)\,C^{4N+2}}{4^{2N+1}[N!\,(N-1)!]^2\,(-\frac{1}{2})!\,[\mu_0(\mu_0+1)]^{N+2}}, \tag{5.26}$$

which gives the leading term of p_i and the first two terms of p_r except in the case where N is an integer. In that case the imaginary term in (5.26) is correct, as is the term $O(C^2)$ in the real part. However, when N is an integer, C_0/D_0 in (5.13) is singular and, although p_i may be obtained from the values for non-integer values of N by letting N tend to an integer, the real part of ϵ obtained from (5.19) does not have a finite limit. The reason is that in this situation p_r contains a term $O(C^6 \log C)$; however, for our purposes this is just a detail and receives no further attention.

This first eigenvalue corresponding to $m = 0$ as given by (5.26) depends on q through μ_0 and N. Results for several values of q have been favourably compared with the corresponding numerical results. Tables have been prepared to illustrate the comparison, and table 1 is an example.† A formula similar to (5.26) has been developed (but not presented here) for the eigenvalues related to the odd eigenfunctions of lowest order. Further consideration of this has been restricted to a comparison with the numerical results for the 'test case' $A = 1$. Again for the first odd and second even eigensolution only the imaginary correction term for the eigenvalues has been found, thus providing a check on the numerical values of p_i calculated for $A = 1$. No further details of the analysis for the higher modes are presented here. We simply observe that (5.26) and all other such results that we have obtained are consistent with the comparison of p and C as deduced from (3.18).

6. Comparison of results

Our numerical results, both for the fundamental and higher modes, all show the same qualitative dependence of p on C. This dependence is illustrated for the scaled growth rate $(-p_i)$ in figure 1: the curves in figure 1 (*a*) are for fundamental modes, the uppermost curve corresponding to $A = 1$ (q complex), the other three corresponding to the physically meaningful values of the swirl parameter, $q = 0.4, 1.0, 1.2$. The curve for the fundamental mode for $A = 1$ is repeated in figure 1 (*b*) (the uppermost curve), the other curves being for the next two lower modes again for $A = 1$. In essence, C^4 (strictly C^4/n^2) shows how far β falls short of $1/q$. It is worth noting that figure 1 (*b*) conveniently illustrates the separation of the modes achieved by the analysis of §3. The curves in figure 1, as well as others not presented here, all have the same essential features and the consistency is both convincing and encouraging.

Now (3.9) results from a limit process in which $C = O(1)$ and the numerical process in §4 is initiated at large values of C and continued as far as possible towards $C = 0$. To start the process with C suitably large, an approximate value of p was obtained from the first *two* terms in (3.11) which is valid for $C \gg 1$ so that good agreement between the asymptotic results for p and our numerical results in the upper range of C is to be expected. It is important to check this agreement but it is more important to compare the numerical results in the lower range of C with the asymptotic results for $C \ll 1$ because for such values of C it was conjectured from the numerical results that $|p_r| \propto C^2$ (approximately) and it was this conjecture that led to the theory for small C in §5. The comparisons for both small and large values of C are contained in the upper and lower sections respectively of a series of tables which list sample values of p. Table 1 for the test case $A = 1$ illustrates the typical features of these tables. The asymptotic values $p_{rAS} + ip_{iAS}$ are found from the first *three* terms of (3.11) for large C and, for the fundamental modes, from (5.26) for small C. The numerical values are listed under p_{rN} and p_{iN}. Table 1 refers to fundamental modes. For both

† The other tables in the series are available on request from the Editorial Office of the *Journal*.

C	p_{rN}	p_{rAS}	p_{iN}	p_{iAS}
0.15	−0.0139	−0.0139	-2.26×10^{-6}	-2.38×10^{-6}
0.20	−0.0248	−0.0248	-1.50×10^{-5}	-1.55×10^{-5}
0.30	−0.0564	−0.0569	-2.24×10^{-4}	-2.11×10^{-4}
0.40	−0.1029	−0.1062	-1.60×10^{-3}	-1.36×10^{-3}
0.80	−0.4199	−0.8645	-2.21×10^{-1}	-1.21×10^{-1}
1.00	−0.418	−0.438	-4.20×10^{-1}	-5.00×10^{-1}
3.00	−0.162	−0.160	-8.33×10^{-1}	-8.33×10^{-1}
5.00	−0.098	−0.100	-9.00×10^{-1}	-9.00×10^{-1}

TABLE 1. Comparison of numerically computed eigenvalues $p_{rN}+\mathrm{i}p_{iN}$ with asymptotic values $p_{rAS}+\mathrm{i}p_{iAS}$ calculated from the results of §5 for the upper sections of the table and from the results of §3 for the lower sections: the case of the first even eigensolution with $A=1$

large and small values of C the agreement throughout is very satisfactory indeed and, as illustrated in table 1, is seen to improve as C decreases through small values in the upper sections of the tables and also as C increases through large values in the lower sections. A crude check on the tabulated values of p for C large shows that the comparison is not inconsistent with the error $O(C^{-3})$ which is expected from the use we have made of (3.11) and (3.16). Comparisons for all cases, for higher modes as well as fundamental modes, are equally convincing. So excellent is the agreement throughout the tables between p_{rAS} and p_{rN} for $C \ll 1$ that no graphical comparison is made. Figure 2 compares p_{iAS} (––––) for $C \ll 1$ with p_{iN} in the cases $A=1$ (the upper pair of curves) and $A=2$ (the lower pair). For the latter case $q=0$. The improving agreement as C decreases is obvious. Thus our numerical results for the eigenvalue dependence on C are not only qualitatively self-consistent over the set of q values considered but are also in excellent agreement with the asymptotic formulae for $C \ll 1$ and $C \gg 1$.

There remains the task of relating our results to those of earlier researchers. Since our results apply to a neighbourhood of the upper neutral point with $n^2(1-\beta q)=O(1)$ as $n\rightarrow\infty$ and $\beta\rightarrow(1/q)_-$, it is appropriate to relate them to the asymptotic theory of Leibovich & Stewartson (1983), for which $(1-\beta q)=O(1)$ as $n\rightarrow\infty$. Under an intermediate-limit process $\beta\rightarrow(1/q)_-$ and $C\rightarrow\infty$ and although we have no formal inner expansion for p in terms of C, and in particular no leading term, such a term, when suitably expressed for large C – as in (3.11) – and after conversion to ω, would be expected to match with the Leibovich & Stewartson result for ω, namely

$$\omega_i = b_0^{\frac{1}{2}} - n^{-\frac{1}{2}}\left(\frac{A_2^2\, b_0}{16\sigma_0^2}\right)^{\frac{1}{4}} + n^{-\frac{3}{2}}\surd\tfrac{1}{2}\,\Gamma_3 + O(n^{-2}), \tag{6.1}$$

$$\omega_r = -n\,\mathrm{e}^{-r_0^2}[\,\beta r_0^2+q-\beta] + n^{-\frac{1}{2}}\surd\tfrac{1}{2}\,\Gamma_1 - n^{-1}\Gamma_2 + n^{-\frac{3}{2}}\surd\tfrac{1}{2}\,\Gamma_3 + O(n^{-2}), \tag{6.2}$$

when β is allowed to approach $1/q$ in these expansions. These are the equations (4.39) and (4.40) given by Leibovich & Stewartson (1983), amended here for typographical errors. Here $A_2=\frac{1}{2}A''(r_0)$ and the other symbols are defined by them, the details being unnecessary for our purposes. Consider the first three terms of (3.11), valid for $C \gg 1$, which for $s=1$ yields

$$p \approx -\mathrm{i} - \tfrac{1}{2}(1-\mathrm{i})\,C^{-1} + (\tfrac{9}{16} - \tfrac{1}{2}A)\,C^{-2}. \tag{6.3}$$

The imaginary part of this is found to match precisely with (6.1) to order $n^{-\frac{1}{2}}$. The real part of (6.3) matches (6.2) up to and including all terms of order n^{-1} except a

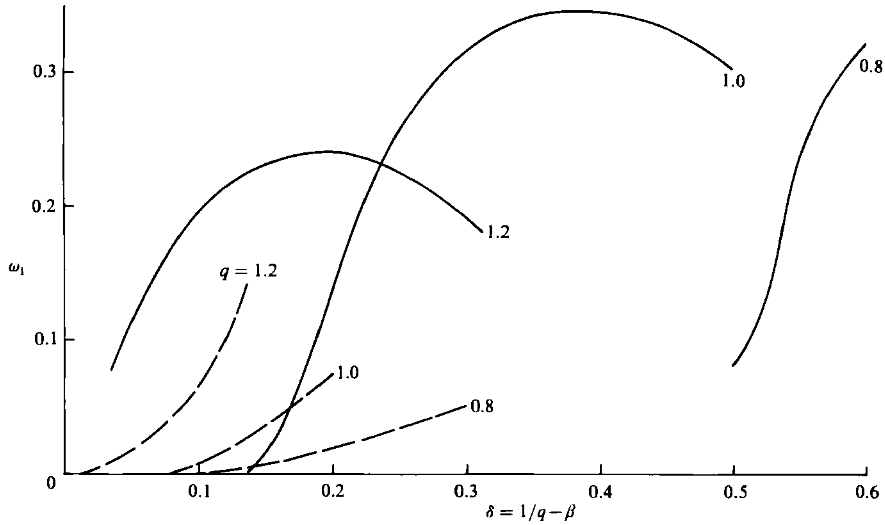

FIGURE 3. Growth rates ω_i for $n = 4$ plotted against $\delta = 1/q - \beta$ for the trailing vortex for $q = 0.8$, 1.0, 1.2: ——, results computed numerically by Leibovich & Stewartson (1983); – – –, our numerical results computed by the method of §4.

contribution $n^{-1}b_1^2/(16A_2 b_0)$ which appears in the expression for A_2 presented by Leibovich & Stewartson. Thus our upper-neutral-point analysis, including the numerical results, is consistent with the earlier asymptotic theory of Leibovich & Stewartson (1983), which it in fact supplements.

The question of a *direct* comparison of our results with earlier *numerical* results does not arise since our results apply with $n \gg 1$ for $\beta \approx 1/q$ where ω_i is small whereas earlier results are for n finite and β is not near $1/q$. The only results previously obtained for small ω_i are those of Leibovich & Stewartson (1983) for the case $n = 4$. Even in this case β_c cannot be predicted with any confidence on the evidence presented by them. A supplementary investigation for finite n is not easy and we attempt to use the results for $n \gg 1$ to assess the likely qualitative structure of near-neutral modes for $n = O(1)$. To this end we formally transform the p_N, C dependence to ω, β dependence using (3.10e) and

$$\omega = nq^{-1}\,\mathrm{e}^{-r_0^2}[1-q^2-r_0^2]-2pr_0\,\mathrm{e}^{-r_0^2}(1-\beta q)^{\frac{1}{2}}, \tag{6.4}$$

which follows from (3.2), (3.4) and (3.8a). The dependence is presented graphically (– – –) in figures 3 and 4 by plotting ω_i and $\beta - q - \omega_r/n = \gamma_r(0)$ against $\delta = 1/q - \beta$. This is done for various values of q in the case $n = 4$. The unbroken curves in figures 3 and 4 have been plotted from tabulated results supplied by Leibovich & Stewartson.

7. Discussion

The investigation of the linearized inviscid stability problem for the trailing vortex is a step towards discovering the role played by hydrodynamic instabilities in nonlinear phenomena, such as vortex breakdown, that occur in concentrated vortex flows. Even when viscous and nonlinear effects are ignored the difficulties in the resulting linearized treatment of unbounded columnar vortices are far from trivial

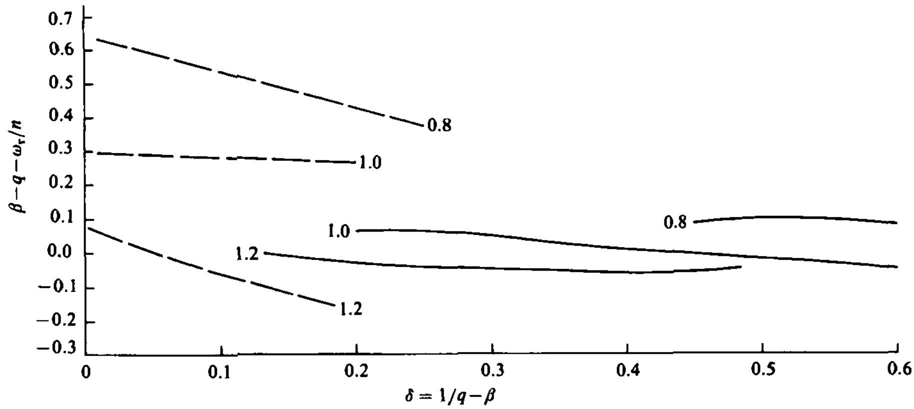

FIGURE 4. Real frequencies of unstable modes corresponding to figure 3.

and any simplifying feature is welcome. When the modes for $\beta V > 0$ (which are more unstable than those for $\beta V < 0$) are investigated, some simplification results by considering large values of n as shown by Leibovich & Stewartson (1983) in their asymptotic analysis of such flows and illustrated by them in the case of the trailing vortex. Even so, their analysis is valid only for the most unstable modes. For these $\frac{1}{2}q < \beta < 1/q$ with β sufficiently distant from the end points of this interval. Their theory for general vortex flows breaks down as $\beta \rightarrow (1/q)_-$ since then $b(r) \rightarrow 0$ for all r.

As Leibovich & Stewartson (1983) point out, in the trailing-vortex flow $a \geqslant 0$ for all β provided $q \geqslant \frac{1}{2}$ and from the criterion for instability,

$$\omega_i \int_0^\infty \left(\frac{a}{|\gamma|^2} + \frac{2\gamma_r b}{|\gamma|^4}\right) |u| \, r \, dr = 0, \tag{7.1}$$

they deduce that $\gamma_r b(r)$ must be negative in some interval of r. Now $b(r) \equiv 0$ if β satisfies $\beta q = 1$, for any specified $q < \surd 2$. In these circumstances $a(r) > 0$ provided $q > 0$, rather than $q > \frac{1}{2}$, and there are then no unstable modes for their trailing-vortex problem for large n. This property continues to apply as $q \rightarrow 0_+$ with $\beta \rightarrow \infty$. There is no evidence from any of the numerical studies that unstable modes occur for $\beta > 1/q$ when $0 < q < \surd 2$ and our numerical work throws no light on this matter. With $\omega_i = 0$, (2.2*a*) is singular at the critical layer. The equation is rendered non-singular by the inclusion of the viscous or time-dependent term. However, in our approach we have attempted to find β_c numerically by effecting the limiting process $\omega_i(\beta) \rightarrow 0$ as $\beta \rightarrow \beta_c$ by an extrapolation towards the $\omega_i = 0$ axis. No earlier attempt to do this has been successful but the stretchings we have introduced for small δ lead to the interesting result $\omega_i \rightarrow 0$ as $\beta \rightarrow (1/q)_-$. Moreover, $\omega_r \rightarrow 0$ in this limit though less rapidly than ω_i. The stretchings used apply even for small values of q with β in the neighbourhood of infinity. The governing equations apply even for $q = 0$ provided we let $A = 2$ in (3.10*d*) since, as $q \rightarrow 0$, $r_0 \rightarrow \infty$ as (3.10*a*) shows. Our results for $q = 0$, both asymptotic and numerical, show that the disturbances approach neutral stability as $C \rightarrow 0$, that is $\beta \rightarrow \infty$, with a similar structure as for $q > 0$. For $q = 0$, the flow is unstable for $0 < \beta < \infty$.

The excellent agreement between our numerical and asymptotic results for both $C \gg 1$ and $C \ll 1$, as well as the agreement between our results and the asymptotic results of Leibovich & Stewartson (1983), strongly support our claim that we have obtained a correct description of the near-neutral structure of the unstable modes near $\beta = 1/q$ for large azimuthal wavenumbers. Our results also provide convincing evidence that the higher modes have a similar structure to the fundamental mode. They become progressively less unstable as the order increases rather than neutrally stable as Duck & Foster (1980) suggest in the case of $n = O(1)$. Because of the lack of data from earlier investigations no comparison with other numerical results is possible, except perhaps for $n = 4$. The need for β steps sufficiently small to avert mode-jumping makes the computations from the full equations very costly for that case and even worse for $n = 5, 6, \ldots$. It is this very difficulty that has led to the present investigation, the results of which give the only available information on the structure of the near-neutral modes. Our results match the asymptotic results of Leibovich & Stewartson (1983) and these in turn do agree more closely with the earlier numerical data as n increases. We infer that our results would also agree more closely with such data if they were extended further towards $\beta = 1/q$. The curves for our results suggest that a 'tail' should develop on the Leibovich–Stewartson curves in figure 3 and such a feature does begin to appear for $q = 0.8$ and 1.0. It is conjectured that for $n = O(1)$ we again have $\omega \to 0$ as $\beta \to (1/q)_-$. A study of the upper neutral points for such n is now under consideration. Another investigation of the modal structure near $\beta = \frac{1}{2}q$ is currently being carried out.

The assistance given by Dr S. N. Brown and Mr P. G. Williams of University College London is greatly appreciated. This research was begun by Keith Stewartson with support from the Fluid Mechanics Program of the National Science Foundation under Grant MEA 79191817. The most important parts of the work were completed at University College London before the untimely death of Keith Stewartson. Of our work together and of our friendship, it is enough to repeat what Keith often said: 'It was fun.'

REFERENCES

Abramowitz, M. & Stegun, I. A. 1968 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. United States National Bureau of Standards.

Batchelor, G. K. 1964 Axial flow in trailing line vortices. *J. Fluid Mech.* **20**, 645–658.

Cebeci, T. & Bradshaw, P. 1977 *Momentum Transfer in Boundary Layers*. McGraw-Hill.

Duck, P. W. & Foster, M. R. 1980 The inviscid stability of a trailing line vortex. *Z. angew. Math. Phys.* **31**, 524–532.

Howard, L. N. & Gupta, A. S. 1962 On the hydrodynamic and hydromagnetic stability of swirling flows. *J. Fluid Mech.* **14**, 463–476.

Leibovich, S. & Stewartson, K. 1983 A sufficient condition for instability of columnar vortices. *J. Fluid Mech.* **126**, 335–356.

Lessen, M., Singh, P. J. & Paillet, F. 1974 The stability of a trailing line vortex. *J. Fluid Mech.* **63**, 753–763.